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THE STABILITY OF UNIFORMLY ACCELERATED MOTIONS†

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Differential equations with quadratic right-hand sides and additional constant terms are considered. Important examples are self-driven gyroscopes and the problem of the motion of a rigid body in an unbounded volume of ideal fluid subject to a force and a torque which are constant in an attached frame of reference. Under certain simple conditions, these equations have solutions that increase linearly with time. In problems of dynamics they describe uniformly accelerated motions of mechanical systems. The stability of such motions is investigated in the first approximation and using bundles of integrals. The general results are used to investigate the stability of uniformly accelerated screw motions of a rigid body in a fluid. © 1999 Elsevier Science Ltd. All rights reserved.

1. STEADY AND UNIFORMLY ACCELERATED MOTIONS

An important part is played in dynamics by equations of the form

$$x^{\bullet}=v(x), \quad x\in R^n \tag{1.1}$$

where the components of the field υ are quadratic forms in the variables $x_1, \ldots, x_n$. Consequently, $\upsilon(\lambda x) = \lambda^2\upsilon(x)$ for all real λ.

One example is provided by the dynamic Euler equations describing the inertial rotation of a rigid body. A more interesting example is the Kirchhoff equations

$$p^{\bullet}=p\times\frac{\partial H}{\partial m}, \quad m^{\bullet}=m\times\frac{\partial H}{\partial m}+p\times\frac{\partial H}{\partial p} \tag{1.2}$$

which defines the motion of a rigid body in an unbounded volume of ideal fluid. Here p is an impulsive force and m is the impulsive momentum of the body in the fluid. The Hamiltonian H (the kinetic energy of the body-plus-fluid system) is a positive definite quadratic form in p and m

$$2H = (Am, m) + 2(Bm, p) + (Cp, p)$$

where A and C are symmetric positive definite matrices.

These two examples are important special cases of the following more general construction. Let us assume that the configuration space of a mechanical system is a Lie group G and its kinetic energy is invariant to left (or right) translations on G. Then, as shown by Poincaré [1], the Lagrange equations have the form (1.1), where $x_1, \ldots, x_n$ are generalized velocities of the system. The dynamic Euler equations correspond to the group $SO(3)$ and the Kirchhoff equations to the group of motions of Euclidean three-space, $E(3)$. Chetayev [2] expressed Poincaré's equations as Hamiltonian equations. For natural reversible systems, the Poincaré–Chetayev equations are quadratic in the phase variables.

Systems with quadratic right-hand sides occur in non-holonomic mechanics. One example is Suslov's problem of the rotation of a rigid body about a fixed point with a non-holonomic constraint: the projection of the angular velocity onto a certain fixed direction in the body vanishes [3].

The motions corresponding to rest points $x = a =$ const of Eqs (1.1) are steady motions of the mechanical system. They are found from the algebraic equations $\upsilon(a) = 0$. As the right-hand sides of (1.1) are homogeneous, the system admits of a whole family of steady motions αu, $\alpha \in R$. For an Euler top these are the permanent rotation of a rigid body about its principal axes of inertia. In the Kirchhoff problem they are screw motions of a body in a fluid (when the velocity of a certain distinguished point and the angular velocity of the body are constant). Rotations about the major and minor axes of inertia are stable while rotations about the middle axis are unstable. The problem of the stability of the screw motions of a rigid body in a fluid was solved by Lyapunov [4].

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Let us complicate the system by adding a force f, which is constant in an attached frame of reference. Equation (1.1) is then replaced by

$$x^{\cdot} = v(x) + f \tag{1.3}$$

Equation (1.3) has a particular solution

$$x(t) = at \tag{1.4}$$

if $f = a =$ const. This equation defines *uniformly accelerated* motion of the system, since the velocities increase in proportion to the time. Since the original system (1.1) has a whole family of steady motions λa, with λ a real parameter, it follows that uniformly accelerated motions certainly exist if the force f is collinear with the steady momentum a.

In the Euler problem, uniformly accelerated motions are possible only when a constant external torque is directed along one of the principal axes of inertia. In that case the Euler equations remain integrable. A detailed analysis of quadratures was carried out by Grammel (see, e.g. [5]). One can consider the more general case in which the torque of the external forces does not depend on the orientation of the body. Grammel called such a top a self-driven top. In a simple but important special case the components of the torque are known functions of time. If such a torque is directed along a principal axis of inertia, one of the motions will be a rotation about that axis at a not necessarily constant angular velocity.

This last observation may be generalized. Let $e = a/|a|$ and $f = \lambda(t)e$. Then Eqs (1.3) have a particular solution

$$x(t) = \mu(t)e \tag{1.5}$$

where μ is a primitive function of λ.

2. STABILITY IN THE FIRST APPROXIMATION

Let $x = a \neq 0$ be a steady-state solution of system (1.1). To investigate its stability we put $x = z + \xi$ and confine our attention to the equations linearized with respect to the increment ξ (equations in variations)

$$\xi^{\cdot} = \Lambda\xi, \quad \Lambda = \frac{\partial v}{\partial x}(a) \tag{2.1}$$

We will show that $\Lambda a = 0$. Since $a \neq 0$ by assumption, one of the eigenvalues of the matrix Λ is always zero. Indeed, by homogeneity, $\upsilon(\alpha a) \equiv 0$ for all $\alpha \in R$. Differentiating this identity with respect to α and then putting $\alpha = 1$, we obtain the required equality.

Thus, one of the eigenvalues of λ is zero. In the stable case, the other eigenvalues either lie in the left half-plane or are pure imaginary and without non-trivial Jordan blocks; multiple roots must have as many linearly independent eigenvalues as their multiplicity. In the linear approximation, the components of the vector ξ are linear combinations with constant coefficients of functions of the following form

$$t^k \exp \mu t, \quad \sin \lambda t, \quad \cos \lambda t \tag{2.2}$$

where $\mu < 0$ and k is any non-negative integer.

Examples of Poincaré–Chetayev equations for which the non-zero eigenvalues of Λ lie in the left half-plane may be found in [6, Chap. 1]. The group G in those examples is solvable but not nilpotent. An analogous property holds for Suslov's non-holonomic system [7].

We now write down the equations in variations for uniformly accelerated motion (1.4), putting $x = at + \eta$ and assuming that the perturbation η is small

$$\eta^{\cdot} = t\Lambda\eta \tag{2.3}$$

where Λ is the operator in (2.1). Thus, Eqs (2.1) and (2.3) differ only in the presence of the factor t. The linear system (2.3) is called a Fuchs system. It is easily solved: if $\xi(t)$ is a solution of (2.1), then $\eta(t) = \xi(t^2/2)$ is a solution of (2.2). Hence, by (2.2), the components of η are linear combinations of functions of the form

$$t^{2k} \exp(\mu t^2/2), \quad \sin(\lambda t^2/2), \quad \cos(\lambda t^2/2)$$

Consequently, the trivial solutions of systems (2.1) and (2.3) are simultaneously stable. If the eigenvalues of the matrix Λ are pure imaginary, except for one which is equal to zero (in which case n must be odd), then the solutions of system (2.3) are superpositions of oscillations whose frequencies increase linearly with time.

We will now show that, if Λ has an eigenvalue in the right half-plane, the solution (1.4) is unstable. Indeed, in terms of the variables $\eta = x - at$, Eqs (1.3) are

$$\eta^{\cdot} = t\Lambda\eta + v(\eta) \tag{2.4}$$

Changing to a new time variable $\tau = t^2/2$ and denoting differentiation with respect to τ by a prime, we obtain

$$\eta' = \Lambda\eta + v(\eta)/\sqrt{2\tau} \tag{2.5}$$

This system satisfies the conditions of the well-known Lyapunov instability theorem [8].

As an illustrative example, let us consider a self-driven gyroscope in which the torque of the forces points along the middle axis of the inertia ellipsoid

$$Ap^{\cdot} + (C-B)qr = 0, \quad Bq^{\cdot} + (A-C)pr = Ba, \quad Cr^{\cdot} + (B-A)pq = 0 \tag{2.6}$$

where $A > B > C$ are the principal axes of inertia, p, q and r are the components of the angular velocity. Equations (2.6) admits of a uniformly accelerated rotation $p = r = 0$, $q = a =$ const. The corresponding 3×3 matrix Λ are eigenvalues

$$0, \ \pm[(A-B)(B-C)/AC]^{1/2}a$$

One of these is always positive. Consequently, the motion is unstable.

An analogous proof yields a similar result on the instability of plane-parallel motion of a body in a fluid with its narrow side forward [9]. Lyapunov's general results [4] on the sufficient conditions for the instability of screw motions of a rigid body in an unbounded volume of ideal fluid carry over to the case of uniformly accelerated motions.

3. THE USE OF QUADRATIC INTEGRALS

In many cases Eqs (1.1) admit of quadratic integrals

$$\Phi(x) = (\Delta x, x)/2 \tag{3.1}$$

which may be used to investigate the stability of steady motions $x = x$. A significant part is played here by the additional assumption

$$\Delta a = 0 \tag{3.2}$$

Put $x = z + \xi$, where ξ is a translation in the direction transverse to the ray of steady motions $x = \alpha a$, $\alpha \in R$ (for example, ξ is orthogonal to a: $(a, \xi) = 0$). Then the quadratic form $\Phi(x) = \Phi(\xi) = (\Delta\xi, \xi)/2$ will also be an integral of the perturbed motion. If the form $\Phi(\xi)$ is positive (or negative) definite, the steady motion $x = a$ is certainly stable with respect to the translation ξ.

The equations of the Euler top (3.2) (in which we must put $a = 0$) admit of a quadratic integral

$$(A-B)Bq^2 + (A-C)Cr^2 \tag{3.3}$$

which satisfies condition (3.2) for permanent rotations about the major axis of inertia: $a = (\alpha, 0, 0)$, $\alpha \neq 0$. Since $A > B > C$, it follows that (3.3) is a positive definite quadratic form in the two variables q and r. Consequently, such motions are stable with respect to the variables q and r. Indeed, steady rotations of a rigid body about the major and minor axes of inertia are stable with respect to all the variables (see, e.g. [10]).

It turns out that if condition (3.2) is satisfied, function (3.1) will be an integral of Eqs (1.3) (in which, of course, $f = a$). Indeed

$$\Phi^{\cdot} = \left(\frac{\partial\Phi}{\partial x}, \ v + a\right) = (\Delta x, a) = (x, \Delta a) = 0$$

because of (3.2). Consequently, the quadratic form $\Phi(\eta) = (\Delta\eta, \eta)/2$ is an integral of the equations of perturbed motion (2.4). Thus, many well-known results concerning the stability of steady motions carry over to the case of uniformly accelerated motions.

For example, uniformly accelerated motions of a top about its major and minor aces of inertia are stable (with respect to translations in transverse directions). An analogous proof yields the stability of the uniformly accelerated plane-parallel motion of a rigid body in a fluid with its broad side forward (see [9]).

Let us apply these observations to Kirchhoff's equations (1.2). Screw motions are found as solutions of the following algebraic system

$$p\times\left(Am+B^T p\right)=0,\quad m\times\left(Am+B^T p\right)+p\times(Bm+Cp)=0$$

Hence constants α and β exist such that

$$Am+B^T p=\alpha p,\quad Bm+Cp=\alpha m+\beta p \tag{3.4}$$

As already observed by Lyapunov [4], these relations are variational in nature. Indeed, Kirchhoff's equations (1.2) admit, besides the energy integral H, of two other quadratic integrals: $\Phi_1 = (m, p)$, $\Phi_2 = (p, p)/2$. The conditions for the function H to be stationary at fixed values of Φ_1 and Φ_2 have the form (3.4). This circumstance enables us to prove the existence of non-trivial solutions of system (3.4). It turns out that for every α three numbers $\beta_1 \leqslant \beta_2 \leqslant \beta_3$, dependent on α exist, for which there are three distinct screw motions of a rigid body with mutually orthogonal screw axes.

Consider the bundle of quadratic integrals

$$\Phi = H-\alpha\Phi_1-\beta\Phi_2 \tag{3.5}$$

defined by the matrix

$$\Delta=\begin{Vmatrix} A & B^T \\ B & C\end{Vmatrix}-\alpha\begin{Vmatrix} 0 & E \\ E & 0\end{Vmatrix}-\beta\begin{Vmatrix} 0 & 0 \\ 0 & E\end{Vmatrix}$$

By (3.4), condition (3.2) is satisfied. Thus, the quadratic form (3.5) is the first integral of the "perturbed" Kirchhoff equations

$$p^{\cdot}=p\times\frac{\partial H}{\partial m}+P,\quad m^{\cdot}=m\times\frac{\partial H}{\partial m}+p\times\frac{\partial H}{\partial p}+M \tag{3.6}$$

where P and M are constant vectors (force and torque) satisfying system (3.4). Equations (3.6) have particular solutions

$$p=Pt,\quad m=Mt \tag{3.7}$$

which may be called uniformly accelerated screw motions.

It has been shown [4] that if $\beta = \beta_1 < \beta_2$, the quadratic form (3.5) is non-negative and vanishes only on the straight line $p = \lambda P$, $m = \lambda M$, $\lambda \in R$. Thus, it is positive definite in any five-dimensional plane which cuts this straight line transversely, and therefore (3.7) are stable solutions of system (3.6).

4. SOME GENERALIZATIONS

The results of Sections 2 and 3 may be extended to solutions of perturbed equations of the form (1.5). Putting $x = \mu(t)e + \eta$, we obtain an equation for the variable η

$$\eta^{\cdot}=\mu\Lambda\eta+v(\eta),\quad \Lambda=\frac{\partial v}{\partial x}(e) \tag{4.1}$$

Suppose the matrix Λ has an eigenvalue with positive real part. Then the trivial solution $\eta = 0$ of system (4.1) is certainly unstable if $\int \mu(t)dt \to \infty$ and the function $1/\mu(t)$ is bounded in some interval $[t_0, +\infty)$.

Indeed, in that interval the function $\mu(t)$ preserves its sign, and we may therefore introduce a new time variable τ by the formula

$$\tau = \int_{t_0}^{t} \mu(u)du \qquad (4.2)$$

and moreover $\tau \to 0$ as $t \to +\infty$. Denoting differentiation with respect to τ by a prime, we can express system (4.1) in the form

$$\eta' = \Lambda\eta + v(\eta)/\mu_* \qquad (4.3)$$

where μ_* is the function μ with the time t replaced by τ in accordance with (4.2). Since by assumption the function μ_*^{-1} is bounded, the equilibrium $\eta = 0$ of system (4.3) (and hence also of (4.1)) is stable by Lyapunov's theorem [8].

The results of Section 3 are applicable to solutions (1.5) without further restrictions on the form of the function $\mu(t)$: if $\Delta e = 0$, the quadratic form (3.1) will be a first integral of Eqs (1.3), where $f = \lambda(t)e$.

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